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THE ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES FOR THE OPERATOR BESSEL EQUATION WITH AN UNBOUNDED OPERATOR COEFFICIENT*

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Abstract. The main goal of this article is to investigate the boundary value problem with an eigenvalue dependent boundary condition for the Bessel operator equation with an unbounded operator coefficient. At first we proved some spectral properties of this problem. Also, the asymptotic behavier of the boundary value problem investigated.

Keywords: separable Hilbert space, self adjoint operator, asymptotics of eigenvalues, discrete spectrum.
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1.Introduction

Spectral theory of operators plays a major role in mathematics and applied sciences. Boundary value problems with an eigenparameter in the boundary conditions are one of the most important fields in the spectral theory of operators. Many researches have been dedicated to studying the spectral properties of the boundary value problems with spectral parameter dependent boundary conditions (see, e.g., Fulton [4], Walter [18]). Various physical applications of such problems can be found in [4]. Note that many problems of mathematical physics, mechanics,theory of partial differential equations, etc are reduced to the study of boundary value problems for operator-differential equations in different spaces. The asymptotic distribution of eigenvalues for boundary-value problems with operator coefficients was first considered by A.G. Kostyuchenko and B.M. Levitan [7] .There followed a lot of works dedicated to the investigation of differential operators' spectrum with operator coefficients. The asymptotic distribution of the eigenvalues of operators defined on the whole space and having a discrete spectrum can be interesting for those who specialize in quantum mechanics. In [17], M.A. Rybak studied the asymptotic behavior of eigenvalues of a boundary value problem with an eigenparameter in the boundary conditions for a second order elliptic operator-differential equation. The symmetry and self-adjointness of the operator associated with this problem were investigated. It was shown that if the considered operator has a discrete spectrum, then the operator associated with

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this problem has a discrete spectrum, too. The asymptotic formula for the eigenvalues of this problem was derived.

The theory of operator-differential equations with unbounded operator-coefficient is a common tool for studying infinite systems of ordinary differential-operators, partial differential equations and integro-differential equations. The main task in this theory is to determine the behavior of the eigenvalues and eigenfunctions of the associated differential operators.

In [1],we considered the spectral problem for the Sturm-Liouville operator in the space $L_2(H,(0,1))$. In this work we consider the spectral problem with the boundary condition which given in general form . The symmetry and self-adjointness of the operator associated with this problem were established and was announced in [13]. The asymptotic formula for the eigenvalues of this problem was derived. If $b=d=1,\ c=0$ in the second con-dition, then the boundary condition takes the form $ay(1)+y'(1)=-\lambda y'(1)$. For this problem the first [14,15] and the second [9,16] trace formulas are established.

In [11,12] the first and the second regularized trace formulas for Sturm-Liouville operators were calculated.

In this paper we consider an operator different from the operators in [2,3] by a boundary condition**.** The main results of this work was announced in [10].

2. Formulation of the problem

H is a separable Hilbert space. A scalar product and a norm in H denoted by $(\cdot,\cdot)$, and $\|\cdot\|$, respectively. $L_2(H,(0,1))$ is a space of the vector functions $y(t)$ such that $\int_0^1 \|y(t)\|^2 dt<\infty$. Let $L_2=L_2(H,(0,1))\oplus H$. Define the scalar product in L_2 as

$$(Y,Z)_{L_2}=\int_0^1 (y(t),z(t))dt+(y_1,z_1) \tag{1}$$

where $Y=\{y(t),y_1\},\ Z=\{z(t),z_1\},\ y(t),z(t)\in L_2(H,(0,1)),\ y_1,z_1\in H.$
Consider the following problem

$$l[y]\equiv -y''(t)+\frac{\nu^2-\frac{1}{4}}{t^2}y(t)+Ay(t)+q(t)y(t)=\lambda y(t) \quad , \quad \nu\ge 1 \tag{2}$$

$$-y(1)=\lambda y'(1) \tag{3}$$

in $L_2(H,(0,1))$ space ,where A is a self-adjoint and positive-definite operator in H ($A>E,\ E$ is an identity operator in H), and has a completely continuous inverse: $A^{-1}\in\sigma_\infty$.

Suppose that the operator-valued function $q(t)$ is weakly measurable, $\| q(t) \|$ is bounded on $[0,1]$.

For $q(t) \equiv 0$ in the space L_2 one can associate with problem (2),(3) a self adjoint operator L_0 defined by

$$D(L_0) = \{Y \in L_2, l[y] \in L_2(H,(0,1)),\ y_1 = y'(1)\}$$

$$L_0 Y = \{-y''(t) + \frac{\nu^2 - \frac{1}{4}}{t^2} y(t) + Ay(t), -y(1)\}.$$

Obviously, for each $y \in D(L_0)$ $y(0) = y'(0) = 0$.

The operator L_0 has a discrete spectrum.

The operator corresponding to the case $q(t) \not\equiv 0$ is denoted by $L = L_0 + Q$, where $Q : Q\{y(t), y'(1)\} = \{q(t)y(t), 0\}$ is a bounded self-adjoint operator in L_2.

The eigenvalues and eigenvectors of the operator A were denoted by $\gamma_1 \le \gamma_2 \le \ldots$ and $\varphi_1, \varphi_2, \ldots$, respectively .

The goal of the paper is to investigate asymptotic eigenvalues distribution of L_0, knowing the asymptote of eigenvalues of the operator A .

3. The asymptotic formula for the eigenvalues of L_0

Suppose that the eigenvalues of the operator A are $\gamma_k \sim ak^\alpha, k \to \infty$, $a > 0, \alpha > 0$. By virtue of the spectral expansion of the operator A, we obtain the following boundary value problem for the coefficients $y_k(t) = (y(t), \varphi_k)$:

$$-y_k''(t) + \frac{\nu^2 - \frac{1}{4}}{t^2} y_k(t) = (\lambda - \gamma_k) y_k(t) \qquad t \in (0,1) \tag{4}$$

$$-y_k(1) = \lambda y_k'(1) \tag{5}$$

The solution to problem (4) from $L_2(0,1)$ is $y_k(t) = \sqrt{t} J_\nu \left(t\sqrt{\lambda - \gamma_k}\right)$. This solution satisfies (5) if and only if

$$J_\nu\left(\sqrt{\lambda - \gamma_k}\right) + \frac{\lambda}{2} J_\nu\left(\sqrt{\lambda - \gamma_k}\right) + \lambda\sqrt{\lambda - \gamma_k} J_\nu'\left(\sqrt{\lambda - \gamma_k}\right) = 0 \tag{6}$$

is true at least for one γ_k $(\lambda \ne \gamma_k)$ .Therefore, the spectrum of the operator L_0 consists of those real values of $\lambda \ne \gamma_k$, such that at least for one k

$$(z^2 + \gamma_k) z J_\nu'(z) + \left(1 + \frac{z^2 + \gamma_k}{2}\right) J_\nu(z) = 0 \tag{7}$$

where $z=\sqrt{\lambda-\gamma_k}$. Then, by using identity $zJ'_\nu(z)=zJ_{\nu-1}(z)-\nu J_\nu(z)$ ([19 p. 56]) in (7), we have

$$\left(z^2+\gamma_k\right)zJ_{\nu-1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1-2\nu)\right)J_\nu(z)=0 \tag{8}$$

Let's find eigenvalues of L_0 , which are less than γ_k . These values are associated with imaginary roots of the equation (8). By taking $z=2i\sqrt{y}$ and using [[19], p. 51]:

$$\sum_{n=0}^{\infty}\frac{y^n}{n!\Gamma(n+\nu+1)}=\frac{J_\nu\left(2i\sqrt{y}\right)}{\left(i\sqrt{y}\right)^\nu}$$

we get

$$\left(-4y+\gamma_k\right)2i\sqrt{y}\left(i\sqrt{y}\right)^{\nu-1}\sum_{n=0}^{\infty}\frac{y^n}{n!\Gamma(n+\nu)}+$$
$$+\left(1+\frac{-4y+\gamma_k}{2}(1-2\nu)\right)\left(i\sqrt{y}\right)^\nu\sum_{n=0}^{\infty}\frac{y^n}{n!\Gamma(n+\nu+1)}=0$$

or

$$2\left(-4y+\gamma_k\right)\sum_{n=0}^{\infty}\frac{y^n}{n!\Gamma(n+\nu)}+$$
$$\left(1+\frac{-4y+\gamma_k}{2}(1-2\nu)\right)\sum_{n=0}^{\infty}\frac{y^n}{n!\Gamma(n+\nu+1)}=$$

$$=\sum_{n=0}^{\infty}\frac{y^n}{n!}\frac{(n+\nu)\left(2\gamma_k-8n^2-4n\nu+6n\right)+1+\frac{\gamma_k}{2}(1-2\nu)}{(n+\nu)\Gamma(n+\nu)}=0 \tag{9}$$

Now find the roots of the following equation

$$(z+\nu)\left(-8z^2-4z\nu+6z+2\gamma_k\right)+1+\frac{\gamma_k}{2}(1-2\nu)=0$$

or

$$z^3+\left(-\frac{3}{4}+\frac{3}{2}\nu\right)z^2+\left(\frac{\nu^2}{2}-\frac{3}{4}\nu-\frac{\gamma_k}{4}\right)z-\frac{\gamma_k\nu+1+\frac{\gamma_k}{2}}{8}=0\,.$$

By substitution $z=x-\dfrac{a}{3}$ [8 pp.234-236] we have

$$z=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{a}{3}$$

where $q=\frac{\nu^2}{8}-\frac{\gamma_k}{8}-\frac{5}{32}$, $p=-\frac{\nu^2}{4}-\frac{\gamma_k}{4}-\frac{3}{16}$, $a=-\frac{3}{4}+\frac{3}{2}\nu$.

From the asymptotics of γ_k we get that the imaginary roots don't exist.

Now, find the asymptotic of those solutions of equation (6) which are greater than γ_k i.e., the real roots of equation (8). By virtue of the asymptotic for a large $|z|$ [[19],p. 222]

$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right),$$

Equation (8) get the form

$$\left(1+\frac{z^2+\gamma_k}{2}(1-2\nu)\right)\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right)-$$

$$-\left(z^2+\gamma_k\right)z\sqrt{\frac{2}{\pi z}}\sin\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right)=0$$

or

$$tg\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)=\frac{1+\frac{z^2+\gamma_k}{2}(1-2\nu)}{\left(z^2+\gamma_k\right)z}\left(1+O\left(\frac{1}{z}\right)\right),$$

$$z-\frac{\nu\pi}{2}-\frac{\pi}{4}=arctg\left(\frac{1+\frac{z^2+\gamma_k}{2}(1-2\nu)}{\left(z^2+\gamma_k\right)z}\left(1+O\left(\frac{1}{z^2}\right)\right)\right)+\pi m=\pi m+O\left(\frac{1}{z^2}\right)$$

where m is a large integer. So in this way we come to the fol-lowing statement.

Lemma 1. For the eigenvalues of L_0 the following asymptotic is true

$$\lambda_{m,k}=\gamma_k+\alpha_m^2 \quad , \ \alpha_m=\left(\pi m+\frac{\nu\pi}{2}+\frac{\pi}{4}\right), m\in Z \quad .$$

Denote the real roots of equation (8) , by $x_{m,k}$ $\left(k=\overline{1,\infty}\right)$.

Let us prove the following two lemmas.

Lemma 2. Equation (8) has only real roots.

Proof. Let α be complex root of the function

$$\left(z^2+\gamma_k\right)zJ_{\nu-1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1-2\nu)\right)J_\nu(z),$$

then $\alpha_0=\overline{\alpha}$ is also a root of this function, since the following series

$$J_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^m}{m!\Gamma(\nu+m+1)}$$

has only real coefficients. From Bessel equation follows [19, c. 531]

$$\int_0^x tJ_\nu(\alpha t)J_\nu(\alpha_0 t)dt=\frac{x}{\alpha^2-\alpha_0^2}\left[J_\nu(\alpha x)\frac{dJ_\nu(\alpha_0 x)}{dx}-J_\nu(\alpha_0 x)\frac{dJ_\nu(\alpha x)}{dx}\right].$$

so, by $\alpha^2\neq\alpha_0^2$ and $J_\nu(\alpha t)=\overline{J_\nu(\alpha_0 t)}$, we get

$$\int_0^1 t\left|J_\nu(\alpha t)\right|^2dt=\frac{1}{\alpha^2-\alpha_0^2}\left[J_\nu(\alpha)\alpha_0J'_\nu(\alpha_0)-J_\nu(\alpha_0)\alpha J'_\nu(\alpha)\right].$$

Take into consideration

$$\left(\alpha^2+\gamma_k\right)\alpha\, J'_\nu(\alpha)=-\left(1+\frac{\alpha^2+\gamma_k}{2}\right)J_\nu(\alpha)\quad,$$

$$\left(\alpha_0^2+\gamma_k\right)\alpha_0\, J'_\nu(\alpha_0)=-\left(1+\frac{\alpha_0^2+\gamma_k}{2}\right)J_\nu(\alpha_0)\quad,$$

we get

$$\int_0^1 t\left|J_\nu(\alpha t)\right|^2dt=\frac{-\dfrac{\left(1+\frac{\alpha_0^2+\gamma_k}{2}\right)J_\nu(\alpha_0)}{\alpha_0^2+\gamma_k}J_\nu(\alpha)+\dfrac{\left(1+\frac{\alpha^2+\gamma_k}{2}\right)J_\nu(\alpha)}{\alpha^2+\gamma_k}J_\nu(\alpha_0)}{\alpha^2-\alpha_0^2}=$$

$$=\frac{-2\left(\alpha^2-\alpha_0^2\right)}{2\left(\alpha^2-\alpha_0^2\right)\left(\alpha^2+\gamma_k\right)\left(\alpha_0^2+\gamma_k\right)}J_\nu(\alpha_0)J_\nu(\alpha)=-\frac{\left|J_\nu(\alpha)\right|^2}{\left(\alpha^2+\gamma_k\right)\left(\alpha_0^2+\gamma_k\right)}<0\,.$$

The integrand on the left hand side is positive, but on the right hand side we get a negative number, which is contradiction.
The lemma is proved.

Let C is a rectangular contour with vertices at the points $\pm iB$, $A_m\pm iB$ which bypasses the origin along small semicircle on the right side of imaginary. Here $A_m=\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}$, B is a large positive number. The following lemma is true.

Lemma 3. For a sufficiently large integer m, the number of zeros of the following function

$$z^{-\nu}\left(\left(z^2+\gamma_k\right)zJ'_\nu(z)+\left(1+\frac{z^2+\gamma_k}{2}\right)J_\nu(z)\right)$$

inside of C is equal to m .

***Proof*.** It is well known that the number of zeroes of the entire function $F(z)$ inside of the closed contour C equals to $\int_C \frac{F'(z)}{F(z)}dz$.

$z^{-\nu}\left(\left(z^2+\gamma_k\right)zJ'_\nu(z)+\left(1+\frac{z^2+\gamma_k}{2}\right)J_\nu(z)\right)$ is an entire function of z, that is why the number of its zeros inside of C equals:

$$\frac{1}{2\pi i}\int_C \frac{\left[z^{-\nu}\left(\left(z^2+\gamma_k\right)zJ'_\nu(z)+\left(1+\frac{z^2+\gamma_k}{2}\right)J_\nu(z)\right)\right]'}{z^{-\nu}\left(\left(z^2+\gamma_k\right)zJ'_\nu(z)+\left(1+\frac{z^2+\gamma_k}{2}\right)J_\nu(z)\right)}dz=$$

$$=\frac{1}{2\pi i}\int_C \frac{\left[z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)\right]'}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz=$$

$$=\frac{1}{2\pi i}\int_C \frac{-\nu z^{-\nu-1}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz+$$

$$+\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ'_{\nu+1}(z)+\left(-3z^2-\gamma_k\right)J_{\nu+1}(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz+$$

$$+\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J'_\nu(z)+z(1+2\nu)J_\nu(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz=$$

$$=\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(\nu\left(z^2+\gamma_k\right)J_{\nu+1}(z)-\frac{\nu}{z}\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz+$$

$$+\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ'_{\nu+1}(z)+\left(-3z^2-\gamma_k\right)J_{\nu+1}(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz+$$

$$+\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J'_\nu(z)+z(1+2\nu)J_\nu(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz=$$

$$=\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(\nu\left(z^2+\gamma_k\right)J_{\nu+1}(z)-\frac{\nu}{z}\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz+$$

$$+\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_\nu(z)+\left(z^2+\gamma_k\right)(\nu+1)J_{\nu+1}(z)+\left(-3z^2-\gamma_k\right)J_{\nu+1}(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz+$$

$$+\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)\frac{\nu}{z}J_\nu(z)-\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_{\nu+1}(z)+z(1+2\nu)J_\nu(z)\right)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz=$$

$$=\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(\nu\left(z^2+\gamma_k\right)-3z^2-\gamma_k+\left(z^2+\gamma_k\right)(\nu+1)-\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)\right)J_{\nu+1}(z)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz+$$

$$+\frac{1}{2\pi i}\int_C \frac{z^{-\nu}\left(-\left(z^2+\gamma_k\right)z+(1+2\nu)z\right)J_\nu(z)}{z^{-\nu}\left(-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)\right)}dz=$$

$$=\frac{1}{2\pi i}\int_C \frac{\left(-z^2-\gamma_k+1+2\nu\right)zJ_\nu(z)+\left(\left(z^2+\gamma_k\right)\left(\nu+\frac{1}{2}\right)-3z^2-\gamma_k+1\right)J_{\nu+1}(z)}{-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)}dz$$

Here we used the identities [19, p. 55]

$zJ'_\nu(z)=\nu J_\nu(z)-zJ_{\nu+1}(z)$, $zJ'_{\nu+1}(z)=zJ_\nu(z)-(\nu+1)J_{\nu+1}(z)$.

As the integrand is an odd function the order of its numerator in the vicinity of zero is $O\left(z^\nu\right)$ and the order of its denominator is $O\left(z^{\nu+1}\right)$, the integral along the left part of contour vanishes. Now, consider the integrals along the remaining three sides of the contour. Note that, on these sides ([19] p.221, p.88)

$$J_\nu(z)=\frac{H_\nu^{(1)}(z)+H_\nu^{(2)}(z)}{2},$$

where $H_\nu^{(1)}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}e^{i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\{1+\eta_{1,\nu}(z)\}$, $H_\nu^{(2)}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}e^{-i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\{1+\eta_{2,\nu}(z)\}$,

$\eta_{1,\nu}(z)$ and $\eta_{2,\nu}(z)$ are of order $O\left(\frac{1}{z}\right)$ for large $|z|$. We get

$$\frac{1}{2\pi i}\int_{iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}^{iB}\frac{\left(-z^2-\gamma_k+1+2\nu\right)zJ_\nu(z)+\left(\left(z^2+\gamma_k\right)\left(\nu+\frac{1}{2}\right)-3z^2-\gamma_k+1\right)J_{\nu+1}(z)}{-\left(z^2+\gamma_k\right)zJ_{\nu+1}(z)+\left(1+\frac{z^2+\gamma_k}{2}(1+2\nu)\right)J_\nu(z)}dz\sim$$

$$\sim\frac{1}{2\pi i}\int_{iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}^{iB}\frac{J_\nu(z)}{J_{\nu+1}(z)}\left(1+O\left(\frac{1}{z}\right)\right)dz=$$

$$\frac{1}{2\pi}\int_{iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}^{iB}\left\{\frac{1+\eta_{2,\nu}(z)}{1+\eta_{2,\nu+1}(z)}\right\}\left[1+O\left(e^{2iz}\right)\right]dz\to\frac{m}{2}+\frac{\nu}{4}-\frac{1}{8}.$$

One can analogously show that the integral along the lower side tends to the same number.

To calculate the integral along the fourth side, we take into consideration the following

$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right),$$

$$J_{\nu+1}(z)=\sqrt{\frac{2}{\pi z}}\sin\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right)$$

we get

$$\frac{1}{2\pi i}\int_{-iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}^{iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}\frac{J_\nu(z)}{J_{\nu+1}(z)}\left(1+O\left(\frac{1}{z}\right)\right)dz=$$

$$\frac{1}{2\pi i}\int_{-iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}^{iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}\frac{cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}{\sin\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\left(1+O\left(\frac{1}{z}\right)\right)dz=$$

$$= \frac{1}{2\pi i} \int_{-iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}^{iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}} ctg\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right)dz \sim$$

$$\frac{1}{2\pi i} \int_{-iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}}^{iB+\pi m+\frac{\nu\pi}{2}-\frac{\pi}{4}} \frac{1}{z}\left(1+O\left(\frac{1}{z}\right)\right)dz \sim \frac{1}{2}$$

Consequently, the limit of the integral along the entire contour is $m+O\left(\frac{1}{m}\right)$ .However,as the integral must be an integer, it should be equal to m . The lemma is proved.

Now, by using the above results, derive the asymptotic formula for the eigenvalue distribution of the operator L_0.

Denote the distribution function of L_0 by $N(\lambda)$ .Then $N(\lambda)=\sum\limits_{\lambda_{m,k}<\lambda} 1$ .So, $N(\lambda)$ is a number of positive integer pairs (m,k) for which $\gamma_k+\alpha_m^2<\lambda$. From the asymptotic of $x_{m,k}$ it follows that one can find a number ε such that for the great values of n

$$(\pi-\varepsilon)m<\alpha_m<(\pi+\varepsilon)m$$

From the asymptotics of γ_k we have

$$(a-\varepsilon)k^\alpha<\gamma_k<(a+\varepsilon)k^\alpha .$$

Hence, we get

$$N_1(\lambda)< N(\lambda)< N_2(\lambda) \qquad (10)$$

Where $N_1(\lambda)$ is the number of the positive integer pairs for which

$$(a-\varepsilon)k^\alpha+(\pi-\varepsilon)^2 m^2<\lambda \qquad (11)$$

$N_2(\lambda)$is the number of the positive integer pairs (m,k) satisfying the inequality

$$(a+\varepsilon)k^\alpha+(\pi+\varepsilon)^2 m^2<\lambda \qquad (12)$$

For $N_2(\lambda)$ as in [[5], Section 3, Lemma 2] we have:

$N_2(\lambda)\le \frac{1}{\pi-\varepsilon}\int_0^{\left(\frac{\lambda}{a-\varepsilon}\right)^{\frac{1}{\alpha}}} \sqrt{\lambda-(a-\varepsilon)x^\alpha}\,dx=\frac{\sqrt{\lambda}}{\pi-\varepsilon}\int_0^{\left(\frac{\lambda}{a-\varepsilon}\right)^{\frac{1}{\alpha}}}\sqrt{1-\frac{(a-\varepsilon)x^\alpha}{\lambda}}dx$ So, by substitution

$$x=\left(\frac{\lambda}{a-\varepsilon}\sin^2 t\right)^{\frac{1}{\alpha}} \quad , \quad dx=\frac{2\sin t\cos t}{\alpha}\left(\frac{\lambda}{a-\varepsilon}\right)^{\frac{1}{\alpha}}\left(\sin^2 t\right)^{\frac{1-\alpha}{\alpha}}dt \quad ,$$

we have

$$N_2(\lambda)\le\frac{2\lambda^{\frac{2+\alpha}{2\alpha}}}{\alpha(\pi-\varepsilon)(a-\varepsilon)^{\frac{1}{\alpha}}}\int_0^{\frac{\pi}{2}}\cos^2 t\sin^{\frac{2}{\alpha}-1}t\,dt=\frac{2}{\alpha(\pi-\varepsilon)(a-\varepsilon)^{\frac{1}{\alpha}}}\gamma\lambda^{\frac{2+\alpha}{2\alpha}} \tag{13}$$

where $\gamma=\int_0^{\frac{\pi}{2}}\cos^2 t\sin^{\frac{2}{\alpha}-1}t\,dt$.

Also for $N_1(\lambda)$ as in [[5], Section 3, Lemma 2] we have:

$$N_1(\lambda)\ge\frac{2\gamma\lambda^{\frac{2+\alpha}{2\alpha}}}{\alpha(\pi+\varepsilon)(a+\varepsilon)^{\frac{1}{\alpha}}}-\left(\frac{\lambda}{a+\varepsilon}\right)^{\frac{1}{\alpha}}-\frac{\sqrt{\lambda}}{\pi+\varepsilon} \tag{14}$$

From (13) and (14) , we have

$$N(\lambda)\sim\frac{2\gamma\lambda^{\frac{2+\alpha}{2\alpha}}}{\alpha(\pi+\varepsilon)(a+\varepsilon)^{\frac{1}{\alpha}}}$$

and consequently

$$\lambda_n(L_0)\sim dn^{\frac{2+\alpha}{2\alpha}} \ , d=\left(\frac{\alpha(\pi+\varepsilon)}{2\gamma}\right)^{\frac{2\alpha}{2+\alpha}}(a+\varepsilon)^{\frac{2}{2+\alpha}}$$

For $\alpha=2$, $N(\lambda,L_0)\sim\frac{\gamma\lambda}{(\pi+\varepsilon)(a+\varepsilon)^{\frac{1}{2}}}$,from which

$$\lambda_n(L_0)\sim dn \ , d=\left(\frac{\gamma}{(\pi+\varepsilon)(a+\varepsilon)^{\frac{1}{2}}}\right)^{-1} .$$

Then, as Q is a bounded operator in L_2 , it follows from the relation for the resolvents of the operators L_0 and L [[6], p. 219]

$$R_\lambda(L)=R_\lambda(L_0)-R_\lambda(L)QR_\lambda(L_0)$$

that the spectrum of L is also discrete. By virtue of the last equality and the properties that hold for singular values of compact operators [[6], pp. 44, 49] as in [[5], Section3, Lemma 2], for the eigenvalues of L, we have

$$\mu_n(L) \sim dn^{\delta}.$$

So, we can state the following theorem:

Theorem1. Let $A = A^* > E$ in H, A^{-1} be compact and eigenvalues of the operator A satisfy the relation $\gamma_k \sim ak^{\alpha}, k \to \infty, a > 0, \alpha > 0$. Then

$$\lambda_n(L_0) \sim \mu_n(L) \sim dn^{\delta}$$

where

$$\delta = \begin{cases} \frac{2\alpha}{\alpha+2}, \alpha > 2, \\ \frac{3}{2}, \alpha < 2, \\ 1, \alpha = 2. \end{cases}$$

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